

Graphs with no induced wheel or antiwheel

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Abstract

A wheel is a graph that consists of a chordless cycle of length at least 4 plus a vertex with at least three neighbors on the cycle. It was shown recently that detecting induced wheels is an NP-complete problem. In contrast, it is shown here that graphs that contain no wheel and no antiwheel have a very simple structure and consequently can be recognized in polynomial time.

Four families of graphs have repeatedly played important roles in structural graph theory recently. They are called *Truemper configurations* as they were first used by Truemper in several theorems [9]. These configurations are called *pyramids*, *prisms*, *thetas* and *wheels*. We will not recall all the definitions, as we do not need all of them here; see Vusković [10] for a very extensive survey on Truemper configurations. It is interesting to know the complexity of deciding whether a graph contains a Truemper configuration of a certain type. The problem is polynomial for pyramids [1]; indeed it is one of the main steps in the polynomial-time recognition algorithm for perfect graphs presented in [1]. On the other hand, the problem is NP-complete for thetas [2] and prisms [7]. Here we will deal only with the fourth Truemper configuration, the wheel. A *wheel* is a graph that consists of a cycle of length at least 4 plus a vertex that has at least three neighbors on the cycle. Diot, Tavenas and Trotignon [3] proved that it is also NP-complete for wheels, and they mention the question of characterizing the graphs that contain no wheel and no antiwheel but leaves it open. This question is solved here with a complete description of the structure of

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these graphs, from which it follows easily that they can be recognized in polynomial (indeed linear) time.

We use the standard graph-theoretic terminology. We let K_n , P_n and C_n respectively denote the complete graph, path and cycle on n vertices, and nF denote the graph with n components, all isomorphic to F . Given a graph family \mathcal{F} , a graph G is \mathcal{F} -free if no induced subgraph of G is isomorphic to any member of \mathcal{F} ; when \mathcal{F} has only one element F we say that G is F -free.

We recall the following simple characterization of P_5 -free bipartite graphs.

Theorem 1 (See [5], [8, Section 2.4]) *Let H be a connected bipartite graph, where $V(H)$ is partitioned into stable sets X and Y . The following conditions are equivalent:*

- H is P_5 -free;
- H is $2K_2$ -free;
- The neighborhoods of any two vertices in X are comparable by inclusion (equivalently, the same holds in Y);
- There is an integer $h > 0$ such that X can be partitioned into non-empty sets X_1, \dots, X_h and Y can be partitioned into non-empty sets Y_1, \dots, Y_h such that for all $i, j \in \{1, \dots, h\}$ a vertex in X_i is adjacent to a vertex in Y_j if and only if $i + j \leq h + 1$.

It follows from Theorem 1 that when H is a P_5 -free connected bipartite graph, with the same notation as in the theorem, then X contains a vertex that is complete to Y , and Y contains a vertex that is complete to X .

Recall that a graph is *split* [4] if its vertex-set can be partitioned into a stable set and a clique. Földes and Hammer [4] gave the following characterization of split graphs.

Theorem 2 ([4]) *A graph is split if and only if it is $\{2K_2, C_4, C_5\}$ -free.*

We define three classes of graphs \mathcal{A} , \mathcal{B} and \mathcal{C} as follows.

Class \mathcal{A} : A graph G is \mathcal{A} if $V(G)$ can be partitioned into two sets $\{a, b, c, d, e\}$ and X such that:

- $\{a, b, c, d\}$ induces a hole with edges ab, bc, cd, da ;
- X is non-empty, induces a clique and is complete to $\{c, d\}$ and anti-complete to $\{a, b\}$;
- e is complete to X , anticomplete to $\{a, b\}$, and has at most one neighbor in $\{c, d\}$.

Class \mathcal{B} : A graph G is in \mathcal{B} if $V(G)$ can be partitioned into four stable sets X, Y, Z, W , with two special vertices $x \in X$ and $y \in Y$, such that:

- $|X| \geq 2$, $|Y| \geq 2$, and $X \cup Y$ induces a connected P_5 -free bipartite graph;
- W is anticomplete to $X \cup Y \cup Z$ (so all vertices of W are isolated in G);
- x is complete to Y and y is complete to X ;
- Z is complete to $\{x, y\}$ and anticomplete to $(X \cup Y) \setminus \{x, y\}$.

Class \mathcal{C} : A graph G is in \mathcal{C} if $V(G)$ can be partitioned in two cliques X and Y of size at least 2 such that the edges between X and Y form a matching of size 2.

Theorem 3 *The following three properties are equivalent:*

- G is (wheel, antiwheel)-free.
- G contains no wheel or antiwheel on at most seven vertices.
- G or \overline{G} is either a 5-hole, a 6-hole, a split graph, or a member of $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$.

Proof. Let F_1 (resp. F_2) be the wheel that consists of a 4-hole plus a vertex adjacent to three (resp. four) vertices of the hole.

Clearly, the first condition of the theorem implies the second. Suppose that G satisfies the third condition. If G or \overline{G} is a 5-hole or a 6-hole, then clearly it does not contain a wheel. If G is a split graph, it contains no hole and consequently no wheel. Suppose that $G \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$. If $G \in \mathcal{A} \cup \mathcal{C}$, it contains only one hole H , of length 4. If $G \in \mathcal{B}$ it may contain many holes, but they all have four vertices, more precisely two vertices from X and two from Y . In all cases, it is easy to see that whenever H is a hole in G , every vertex of $G \setminus H$ has at most two neighbors in H . So no hole of G extends to a wheel.

Now let us prove that the second condition implies the third. Let G be a graph that contains no wheel or antiwheel on at most seven vertices.

First suppose that G contains a 5-hole C . Note that $V(C)$ also induces a 5-hole in \overline{G} . If there is any vertex x in $V(G) \setminus V(C)$, then x has either at least three neighbors in C or three non-neighbors in C , and so $V(C) \cup \{x\}$ induces a wheel in G or in \overline{G} . Thus no such x exists, and G is a 5-hole.

Now suppose that G contains a 6-hole C , with vertices c_1, \dots, c_6 and edges $c_i c_{i+1}$, with subscripts modulo 6. Pick any x in $V(G) \setminus V(C)$. Vertex x has at most two neighbors in C , for otherwise (C, x) is a wheel in G . It

follows that, up to symmetry, $N(x) \cap V(C)$ is equal either to $\{c_1\}$, $\{c_1, c_2\}$, or $\{c_1, c_5\}$, and in that case $\{x, c_1, c_3, c_4, c_6\}$ induces an \overline{F}_1 , or to $\{c_1, c_4\}$, and in that case $\{x, c_2, c_3, c_5, c_6\}$ induces an \overline{F}_2 . Thus no such x exists, and G is a 6-hole.

If G contains a 6-antihole, then the same argument as in the preceding paragraph, applied to \overline{G} , implies that G is a 6-antihole.

Now assume that G contains no 5-hole, no 6-hole and no 6-antihole. We may also assume that G is not a split graph, for otherwise the theorem holds. It follows from Theorem 2 that G contains either a $2K_2$, a C_4 or a C_5 . Since G contains no C_5 , and up to self-complementation, we may therefore assume that G contains a $2K_2$. Let A, B be two disjoint subsets of $V(G)$ such that both A and B are cliques of size at least 2 and A is anticomplete to B . Graph G admits such a pair since we can let A and B be the two cliques of size 2 of a $2K_2$. Choose A and B such that $|A \cup B|$ is maximized. Let $R = V(G) \setminus (A \cup B)$. We observe that:

For every vertex x in R , either:

- (1) • x is complete to A and has a neighbor in B , or vice-versa, or
- x has exactly one non-neighbor in A and one non-neighbor in B .

Indeed, if x has at most one non-neighbor in A and at most one non-neighbor in B , then (1) holds. So suppose, up to symmetry, that x has two non-neighbors a, a' in A . If x has a non-neighbor b in B , then, picking any $b' \in B \setminus b$, we see that $\{x, a, a', b, b'\}$ induces an \overline{F}_1 or \overline{F}_2 (depending on the pair x, b'), a contradiction. So x is complete to B . If x has no neighbor in A , then the pair $A, B \cup \{x\}$ contradicts the choice of A, B . So x has a neighbor in A , and the first item in (1) holds. This proves (1).

Let $A = \{a_1, \dots, a_p\}$, with $p \geq 2$, and let $B = \{b_1, \dots, b_q\}$, with $q \geq 2$. Define the following subsets of R :

- $R_0 = \{x \in R \mid x \text{ is complete to } A \text{ or to } B\}$.
- $R_{i,j} = \{x \in R \mid x \text{ is complete to } (A \cup B) \setminus \{a_i, b_j\} \text{ and anticomplete to } \{a_i, b_j\}\}$, for each $(i, j) \in \{1, \dots, p\} \times \{1, \dots, q\}$.

Clearly these sets are pairwise disjoint, and (1) means that $R = R_0 \cup \bigcup_{i,j} R_{i,j}$.

Say that two vertices x and y of R are *A-comparable* if one of the two sets $N_A(x)$ and $N_A(y)$ contains the other; in the opposite case, say that x and y are *A-incomparable*. Define the same with respect to B .

Suppose that there are two *A-incomparable* vertices x and y in R . Up to relabeling, a_1 is adjacent to x and not to y and a_2 is adjacent to y and not to x . Since each of x and y has a neighbor in B , there is a chordless path P whose endvertices are x and y and whose interior vertices are in B ; and

since B is a clique, the length ℓ of P is equal to 2 or 3. We may assume that if $\ell = 2$ then $P = x-b_1-y$ while if $\ell = 3$ then $P = x-b_1-b_2-y$. Vertices x and y are adjacent, for otherwise $V(P) \cup \{a_1, a_2\}$ induces a 5-hole or a 6-hole. Note that if $p \geq 3$, then x has no neighbor a in $A \setminus \{a_1, a_2\}$, for otherwise $\{a_1, a_2, x, y, a\}$ induces an F_1 or F_2 ; and the same holds for y . So if $p \geq 3$, x and y are anticomplete to $A \setminus \{a_1, a_2\}$ and, by (1), they are complete to B .

Let z be any vertex in $R \setminus \{x, y\}$. Suppose that z is complete to $\{a_1, a_2\}$. Then z is anticomplete to $\{x, y\}$, for otherwise $\{x, y, z, a_1, a_2\}$ induces an F_1 or F_2 . Then z is not adjacent to b_1 , for otherwise either $\{x, y, z, b_1, a_1, a_2\}$ induces a 6-antihole (if $\ell = 2$) or $\{x, y, z, b_1, a_2\}$ induces a 5-hole (if $\ell = 3$). Let b be a neighbor of z in B ; so $b \neq b_1$. Then x is adjacent to b , for otherwise $\{x, z, b, b_1, a_1\}$ induces a 5-hole, and y is adjacent to b , for otherwise $\{x, y, z, b, a_2\}$ induces a 5-hole; but then $\{x, y, z, b, a_1, a_2\}$ induces a 6-antihole. It follows that no vertex of R is complete to $\{a_1, a_2\}$. By the same argument, if $\ell = 3$ then no vertex of R is complete to $\{b_1, b_2\}$, and consequently $R_0 = \emptyset$.

Suppose that $\ell = 3$. The preceding arguments and (1) imply that $R = R_{1,1} \cup R_{1,2} \cup R_{2,1} \cup R_{2,2}$. Note that $x \in R_{2,2}$ and $y \in R_{1,1}$. If $p \geq 3$, then $\{x, y, a_1, a_2, a_3\}$ induces an F_2 . So $p = 2$, and similarly $q = 2$. If there is any vertex u in $R_{1,2}$, then u is adjacent to x , for otherwise $\{u, x, a_1, a_2, b_1\}$ induces a 5-hole, and similarly u is adjacent to y ; but then $\{u, x, y, a_1, a_2\}$ induces an F_1 . So $R_{1,2} = \emptyset$, and similarly $R_{2,1} = \emptyset$. Therefore $V(G) = \{a_1, a_2, b_1, b_2\} \cup R_{1,1} \cup R_{2,2}$. If some vertex u in $R_{1,1}$ is not adjacent to some vertex v in $R_{2,2}$, then $\{u, v, a_1, a_2, b_1, b_2\}$ induces a 6-hole. So $R_{1,1}$ is complete to $R_{2,2}$. If $R_{1,1}$ contains two adjacent vertices u, v , then $\{u, v, x, a_1, a_2\}$ induces an F_1 . So $R_{1,1}$ is a stable set, and similarly $R_{2,2}$ is a stable set. Thus \overline{G} is in class \mathcal{C} .

Now suppose that $\ell = 2$. Let z be any vertex in $R \setminus \{x, y\}$. Suppose that z is anticomplete to $\{a_1, a_2\}$. By (1), z is complete to B and has a neighbor a in $A \setminus \{a_1, a_2\}$. As observed earlier, a is anticomplete to $\{x, y\}$. Then z is adjacent to x , for otherwise $\{x, z, a_1, b_1, a\}$ induces a 5-hole; and similarly z is adjacent to b . But then $\{x, y, z, a_1, a_2, a\}$ induces a 6-antihole. Therefore z has exactly one neighbor in $\{a_1, a_2\}$. Up to symmetry, assume that z is adjacent to a_1 and not to a_2 . If z is adjacent to b_1 , then it is also adjacent to y , for otherwise $\{z, a_1, a_2, y, b_1\}$ induces a 5-hole, and to x , for otherwise $\{z, a_1, x, b_1, y\}$ induces an F_1 ; but then $\{x, y, a_1, a_2, z\}$ induces an F_1 . So z is not adjacent to b_1 , and so $z \in R_{2,1}$. Then z is adjacent to y , for otherwise either $\{z, a_1, a_2, y, b_1, b_2\}$ or $\{z, a_1, a_2, y, b_2\}$ induces a hole, and z is not adjacent to x for otherwise $\{x, y, a_1, a_2, z\}$ induces an F_1 . Then b_2 is adjacent to x , for otherwise $\{x, b_1, b_2, z, a_1\}$ induces a 5-hole, and to y , for

otherwise $\{y, b_1, b_2, z, x\}$ induces an F_1 . But then $\{a_1, z, b_2, x, y\}$ induces an F_1 . This means that $R \setminus \{x, y\} = \emptyset$. If $p \geq 3$, then, as observed earlier, $\{x, y\}$ is anticomplete to $A \setminus \{a_1, a_2\}$ and complete to B . It follows that G is in class \mathcal{C} . Now suppose that $p = 2$. Since x and y are B -comparable, we may assume that $N_B(x) \subseteq N_B(y)$. If B contains two vertices b, b' that are not adjacent to x , then $\{x, a_1, a_2, b, b'\}$ induces an \overline{F}_1 . So B has at most one vertex that is not adjacent to x . If there is such a vertex, then G is in class \mathcal{A} . If there is no such vertex, then G is in class \mathcal{C} .

Now we may assume that any two vertices in R are A -comparable and B -comparable. Since every vertex of R has a neighbor in A , some vertex of A is complete to R . Likewise, some vertex of B is complete to R . So we may assume that a_1 and b_1 are complete to R . If R is not a clique or a stable set, there are three vertices x, y, z in R that induce a subgraph with one or two edges, and $\{a_1, b_1, x, y, z\}$ induces an F_1 or F_2 . Therefore R is a clique or a stable set.

Suppose that R is not a clique. So it is a stable set of size at least 2. A vertex a in $A \setminus \{a_1\}$ cannot have two neighbors x and y in R , for otherwise $\{a_1, x, y, b, a\}$ induces an F_1 . For $k \in \{0, 1\}$, let $A_k = \{u \in A \setminus \{a_1\} \mid u \text{ has } k \text{ neighbors in } R\}$. So $A = \{a_1\} \cup A_0 \cup A_1$. Likewise, let $B_k = \{u \in B \setminus \{b_1\} \mid u \text{ has } k \text{ neighbors in } R\}$. So $B = \{b_1\} \cup B_0 \cup B_1$. Since any two vertices in R are A -comparable, some vertex x in R is complete to A_1 , and $R \setminus \{x\}$ is anticomplete to $A \setminus \{a_1\}$. Likewise, some vertex y in R is complete to B_1 , and $R \setminus \{y\}$ is anticomplete to $B \setminus \{b_1\}$. Suppose that $x = y$. Consider any $z \in R \setminus \{x\}$ (recall that $|R| \geq 2$). Then z is anticomplete to $(A \setminus \{a_1\}) \cup (B \setminus \{b_1\})$, so, by (1), we have $p = q = 2$. Then \overline{G} is in class \mathcal{A} . Now suppose that we cannot choose x and y equal. So both A_1 and B_1 are not empty and we may assume that a_2 is adjacent to x and not to y and b_2 is adjacent to y and not to x . If there is a vertex a_0 in A_0 , then $\{a_0, a_2, x, y, b_2\}$ induces an \overline{F}_1 . So $A_0 = \emptyset$. Likewise $B_0 = \emptyset$. Thus G is in class \mathcal{C} .

Now assume that R is a clique. Since any two vertices of R are A -comparable and B -comparable, there is at most one pair (i, j) such that $R_{i,j} \neq \emptyset$, and since a_1 and b_1 are complete to R , we may assume that $(i, j) = (2, 2)$. Hence $R \setminus R_0 = R_{2,2}$. Let $R^* = \{x \in R \mid x \text{ is complete to } A \cup B\}$, $R_A = \{x \in R \setminus R^* \mid x \text{ is complete to } A\}$ and $R_B = \{x \in R \setminus R^* \mid x \text{ is complete to } B\}$. So $R = R^* \cup R_A \cup R_B \cup R_{2,2}$, and $A \cup R_A$ and $B \cup R_B$ are cliques. Since any two vertices in R are A -comparable and B -comparable, the bipartite subgraph of \overline{G} induced by $A \cup R_A \cup B \cup R_B$ is $2K_2$ -free; moreover, in that graph a_2 is complete to $B \cup R_B$ and b_2 is complete to $A \cup R_A$. It follows that \overline{G} is in class \mathcal{B} (where the four stable sets are $A \cup R_A$, $B \cup R_B$,

$R_{2,2}$ and R^*). This complete to proof of the theorem. \square

The second condition of Theorem 3 implies that deciding whether a graph on n vertices and m edges is (wheel, antiwheel)-free can be done by brute force in time $O(n^7)$. So the problem is polynomially solvable. Actually, one can use the third condition of Theorem 3 to solve the problem in time $O(n^2)$, as follows:

- Testing whether G is a 5-hole or a 6-hole can be done in time $O(n)$.
- Testing whether G is a split graph can be done in time $O(m)$; see [6].
- For each of the classes \mathcal{A} , \mathcal{B} and \mathcal{C} , testing membership in the class can be done in time $O(m)$ directly from the definition of the class (for class \mathcal{B} , using Theorem 1); we omit the details.
- If the above series of tests fails for G , one can run it for \overline{G} in time $O(n^2)$.

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